7.5 More Trigonometric Equations

Sometimes, we use our identities to help solve.
It is helpful if the problem uses only one trigonometric function.

Example 1: Solve.

$1 + \sin x = 2\cos^2 x$

$1 + \sin x = 2(1 - \sin^2 x)$

$\sin x = y$

$1 + \sin x = 2 - 2\sin^2 x$

$-2 + 2\sin^2 x \quad -2 + 2\sin^2 x$

$2\sin^2 x + \sin x - 1 = 0$

$2y^2 + y - 1 = 0$

$(y+1)(2y-1) = 0$

$(\sin x + 1)(2\sin x - 1) = 0$

$\sin x + 1 = 0 \qquad 2\sin x - 1 = 0$

$\sin x = -1 \qquad \sin x = \frac{1}{2}$

$x = \frac{3\pi}{2} \pm 2\pi k \qquad x = \frac{\pi}{6} \pm 2\pi k$

$x = \frac{5\pi}{6} \pm 2\pi k$

Example 2: Solve.

$\sin 2x - \cos x = 0$

$2\sin x\cos x - \cos x = 0$

$\cos x\,(2\sin x - 1) = 0$

$\cos x = 0 \qquad 2\sin x - 1 = 0$

$x = \frac{\pi}{2} \pm \pi k \qquad \sin x = \frac{1}{2}$

$x = \frac{\pi}{6} \pm 2\pi k$

$x = \frac{5\pi}{6} \pm 2\pi k$

Example 3: Solve the equation below in the interval $[0,2\pi)$.

$$(\cos x + 1)^2 = (\sin x)^2$$

hint: square both sides...which means you'll have to check for extraneous solutions

$$\cos^2 x + 2\cos x + 1 = \sin^2 x$$
$$\cos^2 x + 2\cos x + 1 = 1 - \cos^2 x$$
$$+\cos^2 x \quad -1 \quad -1 + \cos^2 x$$
$$2\cos^2 x + 2\cos x = 0$$
$$2\cos x(\cos x + 1) = 0$$

$2\cos x = 0$ $\qquad$ $\cos x + 1 = 0$

$\cos x = 0$ $\qquad$ $\cos x = -1$

$x = \frac{\pi}{2}, \cancel{\frac{3\pi}{2}}$ $\qquad$ $x = \pi$

$\cos\frac{\pi}{2} + 1 \overset{?}{=} \sin\frac{\pi}{2}$

$0 + 1 = 1$ ✓

$\cos\frac{3\pi}{2} + 1 \overset{?}{=} \sin\frac{3\pi}{2}$

$0 + 1 \neq -1$

$\cos\pi + 1 \overset{?}{=} \sin\pi$

$-1 + 1 = 0$ ✓

When solving trigonometric equations that involve functions of multiples of angles, we first solve for the multiple of the angle, then divide to solve for the angle.

Example 4: Solve the equation below in the interval $[0,2\pi)$.

$$2\sin 3x - 1 = 0$$

$$\sin 3x = \frac{1}{2}$$

$\frac{1}{3}\cdot 3x = \frac{\pi}{6}\cdot\frac{1}{3} \pm 2\pi k$ $\qquad$ $\frac{1}{3}\cdot 3x = \frac{5\pi}{6}\cdot\frac{1}{3} \pm 2\pi k$

$x = \frac{\pi}{18} \pm \frac{2}{3}\pi k$ $\qquad$ $x = \frac{5\pi}{18} \pm \frac{2\pi}{3}k$

$x = \frac{\pi}{18}, \frac{13\pi}{18}, \frac{25\pi}{18}$ $\qquad$ $x = \frac{5\pi}{18}, \frac{17\pi}{18}, \frac{29\pi}{18}$

Example 5: Solve the equation below in the interval $[0, 2\pi)$.

$\sqrt{3}\tan\frac{x}{2} - 1 = 0$

$\tan\frac{x}{2} = \frac{\sqrt{3}}{3}$

$2 \cdot \frac{x}{2} = \left(\frac{\pi}{6} + \pi k\right) \cdot 2$

$x = \frac{\pi}{3} + 2\pi k$

$x = \frac{\pi}{3}$

Example 6: Find all solutions of the equation.

$\sin 2x = 2\tan 2x$

$\cos 2x \cdot \sin 2x = \frac{2\sin 2x}{\cos 2x} \cdot \cos 2x$

$\sin 2x \cos 2x = 2\sin 2x$

$\sin 2x \cos 2x - 2\sin 2x = 0$

$\sin 2x\,(\cos 2x - 2) = 0$

$\sin 2x = 0$

$\frac{2x}{2} = \frac{\pi k}{2}$

$x = \frac{\pi k}{2}$

$\cos 2x - 2 = 0$

$\cos 2x \neq 2$

Example 7: Find all solutions of the equation in the interval $[0,2\pi)$.

$3\csc^2 x = 4$

Example 8: Find all solutions of the equation in the interval $[0,2\pi)$.

$\sec x \tan x - \cos x \cot x = \sin x$

$$\frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} - \cos x\cdot\frac{\cos x}{\sin x} = \sin x$$

$$\frac{\sin x}{\cos^2 x} - \frac{\cos^2 x}{\sin x} = \sin x$$

$$\frac{\sin^2 x}{\cos^2 x\sin x} - \frac{\cos^4 x}{\cos^2 x\sin x} = \sin x.$$

$$\cos^2 x\sin x\cdot\frac{\sin^2 x - \cos^4 x}{\cos^2 x\sin x} = \sin x\cdot\frac{\cos^2 x}{\sin x}$$

$$\sin^2 x - \cos^4 x = \cos^2 x\sin^2 x$$

$$0 = \cos^2 x\sin^2 x - \sin^2 x + \cos^4 x$$

$$0 = \cos^2 x(1-\cos^2 x) - (1-\cos^2 x) + \cos^4 x$$

$$0 = \cos^2 x - \cancel{\cos^4 x} - 1 + \cos^2 x + \cancel{\cos^4 x}$$

$$0 = 2\cos^2 x - 1$$

$$\sqrt{\tfrac{1}{2}} = \sqrt{\cos^2 x}$$

$$\pm\frac{\sqrt{2}}{2} = \cos x \qquad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Example 9: Use an addition/subtraction formula to simplify. Then find all solutions in $[0,2\pi)$.

$$\cos x \cos 2x + \sin x \sin 2x = \frac{1}{2}$$

$$\cos(x-2x) = \frac{1}{2}$$

$$\cos(-x) = \frac{1}{2}$$

$$\cos x = \frac{1}{2}$$

$$x = \frac{\pi}{3}, \frac{5\pi}{3}$$

Example 10: Use a double-angle or half-angle formula to solve in $[0,2\pi)$.

$$\tan \frac{x}{2} - \sin x = 0$$

So far, all of the equations we've solved have had solutions like $\frac{\pi}{4}$, $\frac{\pi}{3}$, $\frac{5\pi}{6}$, and so on. We were able to find these solutions from the special values of the trigonometric functions that we've memorized. We now consider equations whose solution requires us to use the inverse trigonometric functions.

Example 11: Solve.

$$\tan^2 x - \tan x - 2 = 0$$

$$(\tan x - 2)(\tan x + 1) = 0$$

$$\tan^{-1}(\tan x) = \tan^{-1}(2) \qquad \tan x = -1$$

$$x \approx 1.11 \pm \pi k \qquad x = \frac{3\pi}{4} \pm \pi k$$

Example 12: Solve. Use a calculator to approximate the solutions in the interval $[0, 2\pi)$, correct to 5 decimals.

$$3 \sin x - 2 = 0$$

$$\sin^{-1}(\sin x) = \sin^{-1}\left(\frac{2}{3}\right)$$

$$x \approx 0.72973, 2.41027$$

Example 13: a) Find all solutions of the equation.
b) Use a calculator to solve the equation in the interval $[0,2\pi)$, correct to 5 decimal places.

$$\frac{3 \sin x}{\cos x} = \frac{7 \cos x}{\cos x}$$

$$3 \tan x = 7$$

$$\tan^{-1}(\tan x) = \tan^{-1}\left(\frac{7}{3}\right)$$

$$x \approx 1.16590,\ 4.30590$$

Example 14: a) Find all solutions of the equation.
b) Use a calculator to solve the equation in the interval $[0,2\pi)$, correct to 5 decimal places.

$2 \tan x = 13$