$\color{red}{9.5}$ Hyperbolas

Standard Form of a Horizontal Hyperbola
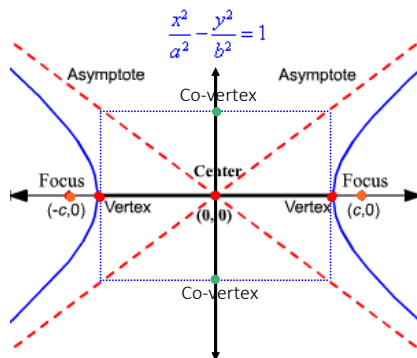
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Center: (0,0)

a = distance from center to vertex = ALWAYS the 1st denominator

b = distance from center to co-vertex = ALWAYS the 2nd denominator

c = distance from center to a focus point

$c^2 = a^2 + b^2$

Vertices: (±a,0)

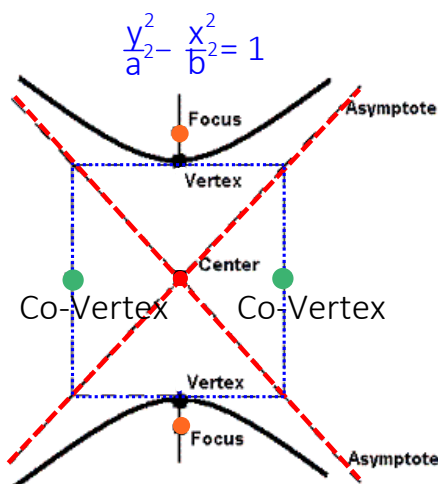Co-vertices: (0,±b)

Foci: (±c,0)

The asymptotes "guide the shape" of the hyperbola. They are two lines, and therefore, must be written in equation form.

Asymptotes: $y = \pm \dfrac{b}{a}x$

Standard Form of a Vertical Hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

Center: (0,0)

a = distance from center to vertex = ALWAYS the 1st denominator

b = distance from center to co-vertex = ALWAYS the 2nd denominator

c = distance from center to a focus point

$c^2 = a^2 + b^2$

Vertices: (0,±a)

Co-vertices: (±b,0)

Foci: (0,±c)

The asymptotes "guide the shape" of the hyperbola. They are two lines, and therefore, must be written in equation form.

Asymptotes: $y = \pm \dfrac{a}{b}x$

Steps for Sketching a Hyperbola
1. Sketch the asymptotes.
2. Plot the vertices.
3. To sketch the graph, start at a vertex and sketch one branch of the hyperbola, approaching the asymptotes. Then sketch the other branch.

Example 1
Find the center, vertices, co-vertices, foci, & asymptotes.
Then graph. $a = 6$ $b = 2$ $c = 2\sqrt{10}$ $36 + 4 = c^2$
                                                              $40 = c^2$

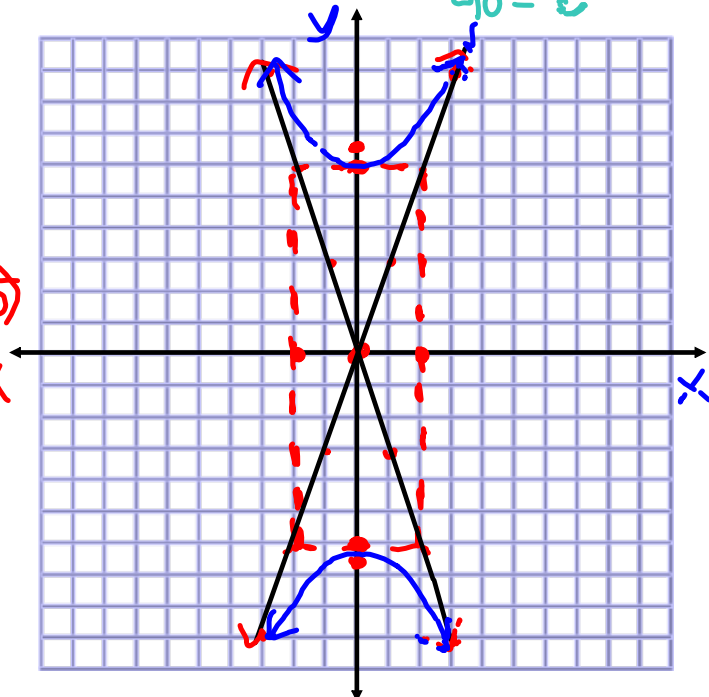$$\frac{y^2}{36} - \frac{x^2}{4} = 1$$

center $(0,0)$
vertices $(0,6), (0,-6)$
co-vert $(-2,0), (2,0)$
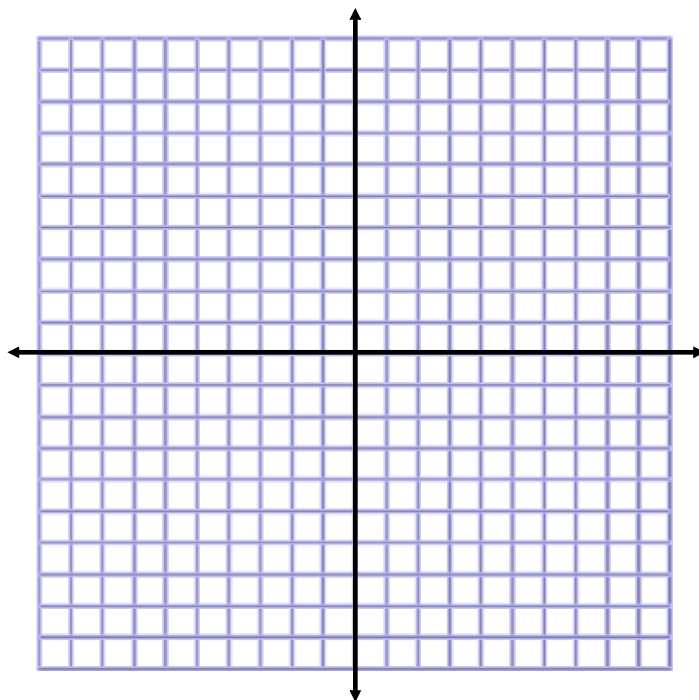foci $(0, 2\sqrt{10}), (0, -2\sqrt{10})$
asymptotes: $y = \pm 3x$

Example 2
Find the center, vertices, co-vertices, foci, & asymptotes.
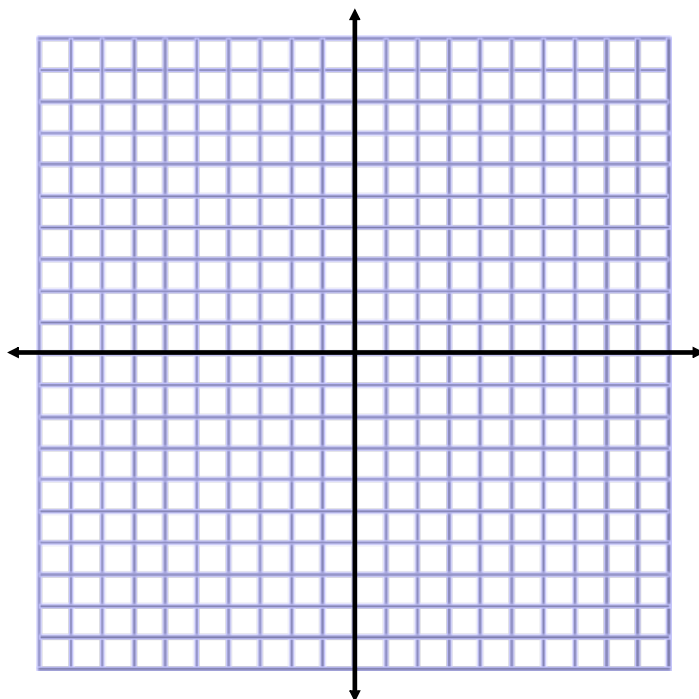Then graph.
$9x^2 - 16y^2 = 144$

Example 3
Find the center, vertices, co-vertices, foci, & asymptotes.
Then graph.
$x^2 - 9y^2 + 9 = 0$

$$\frac{x^2}{-9} - \frac{9y^2}{-9} = \frac{-9}{-9}$$

$$-\frac{x^2}{9} + \frac{y^2}{1} = 1$$

$$\frac{y^2}{1} - \frac{x^2}{9} = 1$$

Example 4

Find the equation of the hyperbola with vertices (±3,0) and foci (±4,0). Then sketch the graph.
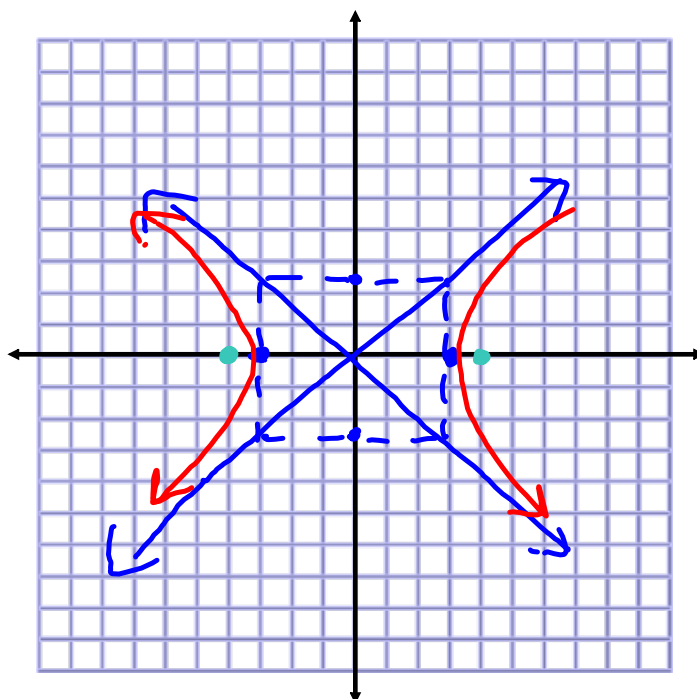
$$a=3$$

$$c=4$$

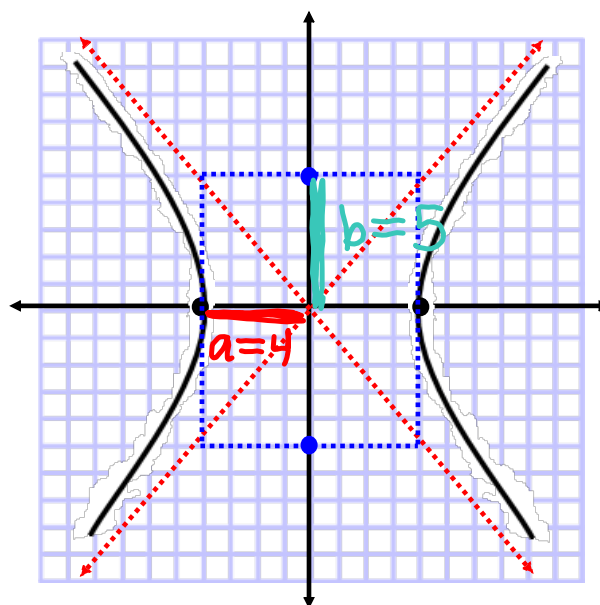$$a^2 + b^2 = c^2$$
$$9 + b^2 = 16$$
$$b^2 = 7$$
$$b = \sqrt{7}$$

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

Example 5

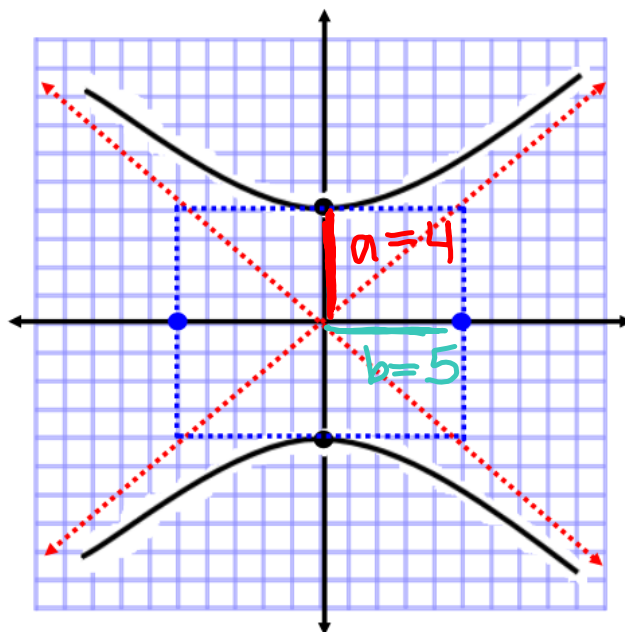Find the equation for the hyperbola whose graph is shown.

$$\frac{x^2}{16} - \frac{y^2}{25} = 1$$

$$b = 5$$

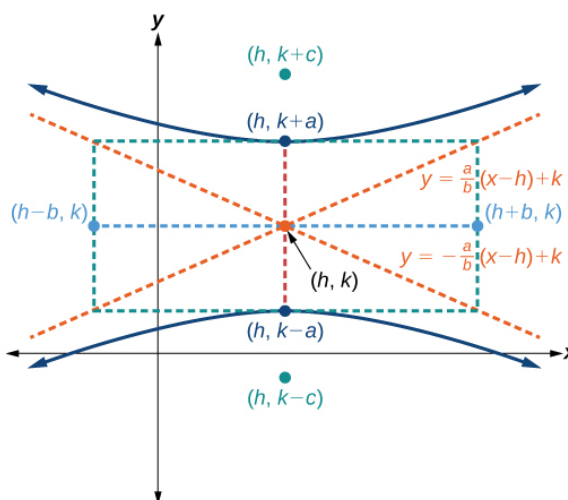$$a = 4$$

Example 6

Find the equation for the hyperbola whose graph is shown.

$$\frac{y^2}{16} - \frac{x^2}{25} = 1$$

$a = 4$

$b = 5$

Standard Equation of a Translated Hyperbola

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1 \qquad\qquad \frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$$

$y = -\frac{b}{a}(x-h)+k$

$(h, k+b)$

$(h-c, k)$ $(h+c, k)$

(h, k)

$(h-a, k)$ $(h+a, k)$

$(h, k-b)$

$y = \frac{b}{a}(x-h)+k$

$(h, k+c)$

$(h, k+a)$

$(h-b, k)$

$y = \frac{a}{b}(x-h)+k$

$(h+b, k)$

$y = -\frac{a}{b}(x-h)+k$

(h, k)

$(h, k-a)$

$(h, k-c)$
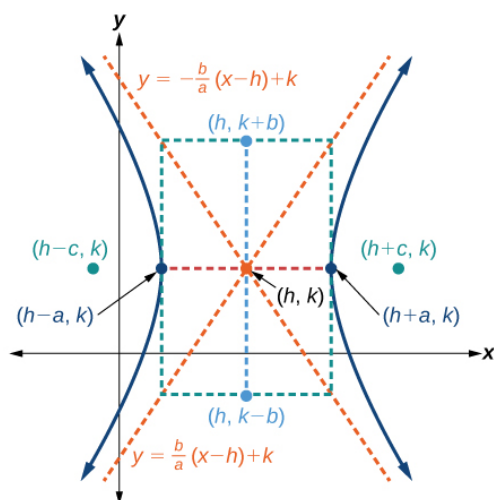
Example 7

Find the center, vertices, co-vertices, foci, & asymptotes.
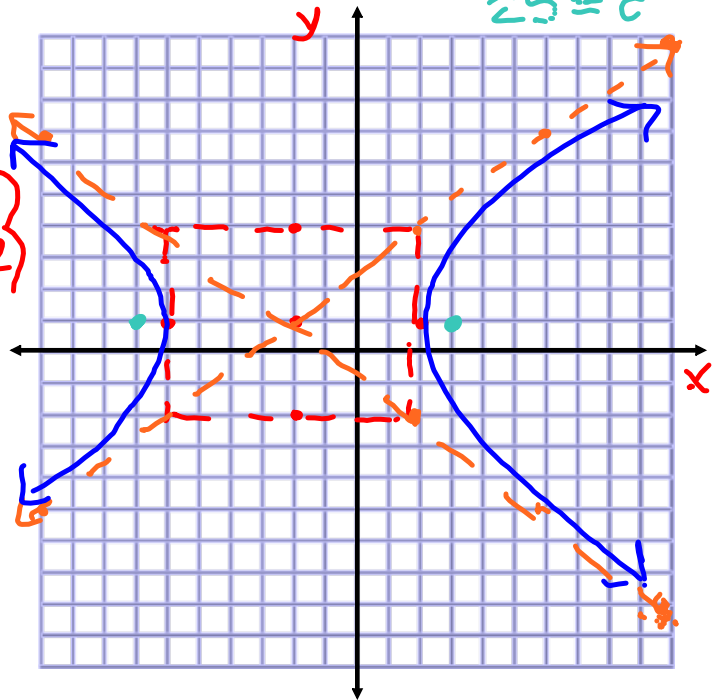Then graph. $a=4$ $b=3$ $c=5$ $16+9=c^2$

$$\frac{(x+2)^2}{16} - \frac{(y-1)^2}{9} = 1$$

$$25 = c^2$$

Center $(-2, 1)$

Vertices $(-6, 1), (2, 1)$

Co-vert $(-2, 4), (-2, -2)$

foci $(-7, 1), (3, 1)$

asym: $m = \pm \frac{3}{4}$

Example 8

Find the center, vertices, co-vertices, foci, & asymptotes.
Then graph. $a=2$ $b=6$ $c=2\sqrt{10}$ $36+4=c^2$
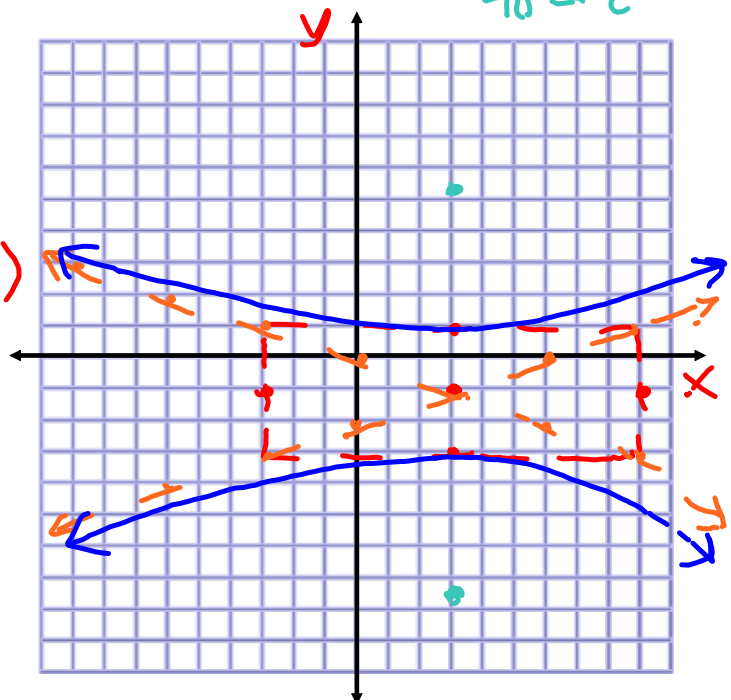
$$\frac{(y+1)^2}{4} - \frac{(x-3)^2}{36} = 1$$

$$40 = c^2$$

Center $(3, -1)$

vert $(3, 1), (3, -3)$

Co-vert $(-3, -1), (9, -1)$

foci $(3, -1 \pm 2\sqrt{10})$

asym: $m = \pm \frac{1}{3}$

Example 9
Write the standard equation for the hyperbola with given characteristics:

opens L/R

a) vertices at (-5,0) and (5,0); co-vertices at (0,-6) and (0,6)

$a = 5$ $b = 6$

$$\frac{x^2}{25} - \frac{y^2}{36} = 1$$

opens U/D

b) vertices at (0,-7) and (0,7); foci at (0,-9) and (0,9)

$a = 7$ $c = 9$

$$\frac{y^2}{49} - \frac{x^2}{32} = 1$$

$a^2 + b^2 = c^2$

$49 + b^2 = 81$

$b^2 = 32$

Example 10
Write the standard equation for the hyperbola with given characteristics:

a) foci at (-1,1) and (5,1); vertices at (0,1) and (4,1)

b) vertices at (3,-1) and (3,5); foci at (3,-3) and (3,7)

Example 11

Write the standard equation for the hyperbola. Give the coordinates of the center, vertices, co-vertices, and foci.

$-2x^2 + y^2 + 4x + 6y + 3 = 0$

$4 + 2 = c^2$

$6 = c^2$

$$y^2 + 6y \qquad -2x^2 + 4x \qquad = -3$$

$$y^2 + 6y + 9 \quad -2(x^2 - 2x + 1) = -3 + 9 - 2$$

$\frac{1}{2}(6) = 3$ \qquad $\frac{1}{2}(-2) = -1$

$3^2 = 9$ \qquad $(-1)^2 = 1$

$$\frac{(y+3)^2}{4} - \frac{2(x-1)^2}{4} = \frac{4}{4}$$

$a = 2$

$b = \sqrt{2}$

$c = \sqrt{6}$

$$\frac{(y+3)^2}{4} - \frac{(x-1)^2}{2} = 1$$

center $(1, -3)$ \qquad vert $(1, -1), (1, -5)$

co-vert $(1 \pm \sqrt{2}, -3)$

foci $(1, -3 \pm \sqrt{6})$

Example 12

Write the standard equation for the hyperbola. Give the coordinates of the center, vertices, co-vertices, and foci.

$4x^2 - 25y^2 - 8x + 100y - 196 = 0$

$$4x^2 - 8x \qquad -25y^2 + 100y \qquad = 196$$

$$4(x^2 - 2x + 1) - 25(y^2 - 4y + 4) = 196 + 4 - 100$$

$\frac{1}{2}(-2) = -1$ \qquad $\frac{1}{2}(-4) = -2$

$(-1)^2 = 1$ \qquad $(-2)^2 = 4$

$$\frac{4(x-1)^2}{100} - \frac{25(y-2)^2}{100} = \frac{100}{100}$$

$a = 5 \quad b = 2$

$c = \sqrt{29}$

$$\frac{(x-1)^2}{25} - \frac{(y-2)^2}{4} = 1$$

$25 + 4 = c^2$

center $(1, 2)$

vertices $(-4, 2), (6, 2)$

co-vert $(1, 4), (1, 0)$

foci $(1 \pm \sqrt{29}, 2)$